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Asymptotic Behaviour of Orthogonal Polynomials on the Unit Circle with Asymptotically Periodic Reflection Coefficients

II. Weak Asymptotics[1]

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In this paper we study orthogonal polynomials with asymptotically periodic reflection coefficients. It's known that the support of the orthogonality measure of such polynomials consists of several arcs. We are mainly interested in the asymptotic behaviour on the support and derive weak convergence results for the orthogonal polynomials and also for the Christoffel function. © 2000 Academic Press

Key Words: orthogonal polynomials; unit circle; arcs; weak asymptotics; asymptotically periodic reflection coefficients; reproducing kernel function.

1. INTRODUCTION AND NOTATION

Let $P_n(z, \sigma)$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, be the monic orthogonal polynomials on the unit circle with respect to the probability measure $\sigma/2\pi$. That is, σ is a finite nonnegative Borel measure with $\operatorname{supp}(\sigma) \subseteq [0, 2\pi]$ and $\sigma([0, 2\pi]) = 2\pi$. We further assume that the support of σ is an infinite set.

It is well known that the P_n's satisfy a recurrence relation of the form

$$P_{n+1}(z, \sigma) = zP_n(z, \sigma) + a_n P_n^*(z, \sigma), \qquad n \in \mathbb{N}_0, \quad P_0(z, \sigma) = 1, \quad (1.1)$$

where the parameters $a_n := a_n(\sigma) := \overline{P_{n+1}(0, \sigma)}$ are called reflection coefficients and satisfy $|a_n| < 1$. $P_n^*(z, \sigma) := z^n \overline{P_n(1/\bar{z}, \sigma)}$ denotes the

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so-called reversed polynomial. The orthogonality property can be written as

$$\frac{1}{2\pi}\int_0^{2\pi} P_n(e^{i\varphi},\sigma)\,\overline{P_m(e^{i\varphi},\sigma)}\,d\sigma(\varphi)=\delta_{nm}d_n, \tag{1.2}$$

where $d_n := \prod_{j=0}^{n-1}(1-|a_j|^2)$. As an immediate consequence, by the normalization

$$\Phi_n(z,\sigma) := \frac{P_n(z,\sigma)}{\sqrt{d_n}} = \frac{z^n}{\sqrt{d_n}} + \cdots, \qquad n\in\mathbb{N}_0, \tag{1.3}$$

we get the unique orthonormal polynomials with respect to σ with positive leading coefficient:

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi_n(e^{i\varphi},\sigma)\,\overline{\Phi_m(e^{i\varphi},\sigma)}\,d\sigma(\varphi)=\delta_{nm}.$$

In this paper we study orthogonal polynomials whose reflection coefficients are asymptotically periodic; i.e., there exist values $a_0^0, ..., a_{N-1}^0$, $N\in\mathbb{N}$, $|a_j^0|<1$, such that

$$\lim_{\nu\to\infty} a_{\nu N+j}=a_j^0 \qquad \text{for} \quad j=0, 1, ..., N-1. \tag{1.4}$$

Let $\{a_n^0\}$ denote the periodically extended sequence, i.e.,

$$a_n^0 = a_{n+N}^0 \qquad \text{for} \quad n\in\mathbb{N}_0.$$

A crucial role will play the monic polynomials $P_n(z,\sigma_0)$, $n\in\mathbb{N}_0$, generated recursively by the periodic sequence $\{a_n^0\}$

$$P_{n+1}(z,\sigma_0)=zP_n(z,\sigma_0)+a_n^0P_n^*(z,\sigma_0), \qquad n\in\mathbb{N}_0, \quad P_0(z,\sigma_0)=1.$$

The $P_n(z,\sigma_0)$'s are orthogonal polynomials with respect to a measure σ_0, as indicated by the notation, which can be given even explicitly (see [5, 15]). The corresponding orthonormal polynomials are denoted by $\Phi_n(z,\sigma_0) := P_n(z,\sigma_0)/\sqrt{d_n^0}$, $d_n^0 := \prod_{j=0}^{n-1}(1-|a_j^0|^2)$.

In [16, 17] the authors have studied the asymptotic behaviour of the "asymptotically periodic" orthogonal polynomials $\Phi_n(z,\sigma)$ for $n\to\infty$ outside the support of the measure of orthogonality. Ratio asymptotics of such orthogonal polynomials have been given recently by Barrios and López [1]. In this paper we are again interested in the asymptotic behaviour, but on the support of the orthogonality measure. We show that the orthonormal polynomials $\Phi_n(z, \sigma)$ converge weakly on the support. As a byproduct we obtain an extension of a result of Máté, Nevai, and Totik [10] to the asymptotically periodic case. Let us recall that in [10, Theorem 5] it is shown that

$$\lim_{n \to \infty} |\Phi_n(e^{i\varphi}, \sigma)|^2 \, d\sigma(\varphi) = d\varphi \qquad \text{in the weak-} * \text{ sense,} \tag{1.5}$$

whenever the reflection coefficients $a_n = P_{n+1}(0, \sigma)$ satisfy

$$\lim_{n \to \infty} a_n = 0,$$

i.e., in our notation $N = 1$ and $a_0^0 = 0$. Using our extension of this result we also derive asymptotics for the reproducing kernel function, i.e., for the Christoffel function $K_n(e^{i\varphi}, e^{i\varphi}; \sigma)$, on the support of the measure of orthogonality. The statements are presented in Section 3. All the proofs are given in Section 4. But first of all, in Section 2, we will state some basic properties of the "periodic" orthogonal polynomials $P_n(z, \sigma_0)$.

Let us finish this section with some more definitions. The monic polynomial of the second kind of $P_n(z, \sigma)$, where $\{P_n(z, \sigma)\}$ satisfies (1.1), is given recursively by

$$\Omega_{n+1}(z, \sigma) := z\Omega_n(z, \sigma) - a_n \Omega_n^*(z, \sigma), \qquad \Omega_0(z, \sigma) := 1.$$

Furthermore, the monic associated polynomials $\{P_n^{(k)}(z, \sigma)\}_{n \in \mathbb{N}_0}$, resp. the monic associated polynomials of the second kind $\{\Omega_n^{(k)}(z, \sigma)\}_{n \in \mathbb{N}_0}$, of order k, $k \in \mathbb{N}_0$, are given by the shifted recurrence formula

$$P_{n+1}^{(k)}(z, \sigma) := zP_n^{(k)}(z, \sigma) + a_{n+k} P_n^{(k)*}(z, \sigma), \qquad P_0^{(k)}(z, \sigma) := 1$$

$$\Omega_{n+1}^{(k)}(z, \sigma) := z\Omega_n^{(k)}(z, \sigma) - a_{n+k} \Omega_n^{(k)*}(z, \sigma), \qquad \Omega_0^{(k)}(z, \sigma) := 1.$$

Of course, these polynomials are again orthogonal on the unit circle. The orthonormalized associated polynomials are defined by

$$\Phi_n^{(k)}(z, \sigma) := \frac{P_n^{(k)}(z, \sigma)}{\sqrt{d_n^{(k)}}}, \qquad \Psi_n^{(k)}(z, \sigma) := \frac{\Omega_n^{(k)}(z, \sigma)}{\sqrt{d_n^{(k)}}},$$

where $d_n^{(k)} := \prod_{j=0}^{n-1} (1 - |a_{j+k}|^2)$. Naturally, the associated polynomials have asymptotically periodic reflection coefficients, if (1.4) is satisfied. Many properties of the associated polynomials can be found in [13]. Among others we have the relations

$$\begin{aligned}
2P_{n+k}(z,\sigma) &= (P_k(z,\sigma)+P_k^*(z,\sigma))\,P_n^{(k)}(z,\sigma)\\
&\quad +(P_k(z,\sigma)-P_k^*(z,\sigma))\,\Omega_n^{(k)}(z,\sigma)\\
2\Omega_{n+k}(z,\sigma) &= (\Omega_k(z,\sigma)+\Omega_k^*(z,\sigma))\,\Omega_n^{(k)}(z,\sigma)\\
&\quad +(\Omega_k(z,\sigma)-\Omega_k^*(z,\sigma))\,P_n^{(k)}(z,\sigma)\\
2P_n^{(k)}(z,\sigma) &= \frac{1}{d_k z^k}\,[P_{n+k}(z,\sigma)(\Omega_k(z,\sigma)+\Omega_k^*(z,\sigma))\\
&\quad -\Omega_{n+k}(z,\sigma)(P_k(z,\sigma)-P_k^*(z,\sigma))]\\
2\Omega_n^{(k)}(z,\sigma) &= \frac{1}{d_k z^k}\,[\Omega_{n+k}(z,\sigma)(P_k(z,\sigma)+P_k^*(z,\sigma))\\
&\quad -P_{n+k}(z,\sigma)(\Omega_k(z,\sigma)-\Omega_k^*(z,\sigma))],
\end{aligned} \tag{1.6}$$

$n,\ k\in\mathbb{N}_0$.

2. BASIC PROPERTIES OF ORTHOGONAL POLYNOMIALS WITH PERIODIC REFLECTION COEFFICIENTS

Orthogonal polynomials $P_n(z,\sigma_0)$ with periodic reflection coefficients

$$a_n^0 = a_{n+N}^0, \qquad n\in\mathbb{N}_0, \quad N\in\mathbb{N} \text{ fixed},$$

have been studied by Geronimus [4, 5] and later also by the authors [15]. Let us give some basic facts about the "periodic" measure σ_0, which are needed in what follows.

It is known that the support of σ_0 consists of l, $l\leqslant N$, disjoint subintervals of $[0, 2\pi]$ and at most of a finite number of points outside the intervals. Let us denote these intervals by

$$E_l := \bigcup_{j=1}^{l} [\varphi_{2j-1}, \varphi_{2j}], \tag{2.1}$$

where the φ_k's, $k=1, ..., 2l$, are pairwise distinct. For the corresponding arcs on the unit circle we write

$$\Gamma_{E_l} := \{e^{i\varphi}\colon \varphi\in E_l\}.$$

The set E_l and the measure σ_0 can completely be described by the orthogonal polynomials $P_n(z, \sigma_0)$ in the following way (see [15, Sects. 2, 4, and 5]). Put

$$L := 2\left(\prod_{j=0}^{N-1} (1-|a_j^0|^2)\right)^{1/2} = 2\sqrt{d_N^0} \tag{2.2}$$

and

$$\begin{aligned} \mathscr{T}(z) &:= \tfrac{1}{2}(P_N(z, \sigma_0) + \Omega_N(z, \sigma_0) + P_N^*(z, \sigma_0) + \Omega_N^*(z, \sigma_0)) \\ &= z^N + \cdots \\ \mathfrak{R}(z) &:= \mathscr{T}^2(z) - L^2 z^N = z^{2N} + \cdots. \end{aligned} \tag{2.3}$$

Then $\mathscr{T}$ and $\mathfrak{R}$ are selfreversed polynomials which have all their zeros on $|z| = 1$. In particular, $\mathfrak{R}$ has a simple zero at each end-point $e^{i\varphi_j}$, $j = 1, ..., 2l$, of the arcs and exactly $N - l$ double zeros $e^{i\psi_1}, ..., e^{i\psi_{N-l}}$ in $\{e^{i\varphi}: \varphi \in \operatorname{int} E_l\}$. Thus $\mathfrak{R}$ is of the form

$$\mathfrak{R}(z) = R(z)\, \mathscr{U}^2(z),$$

where $R(z) = z^{2l} + \cdots$ and $\mathscr{U}(z) = z^{N-l} + \cdots$ are selfreversed polynomials which vanish exactly at the $e^{i\varphi_j}$'s and $e^{i\psi_j}$'s, respectively. Recall that a polynomial Q is called selfreversed if $Q(z) = \mu Q^*(z)$, where $|\mu| = 1$. Note, if Q is a selfreversed polynomial then $e^{-i(\partial Q/2)\varphi} \mu^{1/2} Q(e^{i\varphi})$ is a real trigonometric polynomial of degree $\partial Q/2$. Now, the set E_l defined in (2.1) can be expressed with the aid of the polynomials R and $\mathscr{T}$, respectively, by

$$E_l = \{\varphi \in [0, 2\pi]: e^{-il\varphi} R(e^{i\varphi}) \leqslant 0\} = \{\varphi \in [0, 2\pi]: |\mathscr{T}(e^{i\varphi})| \leqslant L\}.$$

Furthermore, the absolutely continuous part f_0 of σ_0 is given explicitly in terms of the corresponding orthogonal polynomials by

$$f_0(\varphi) = \begin{cases} \left|\dfrac{\sqrt{R(e^{i\varphi})}}{V(e^{i\varphi})\, A(e^{i\varphi})}\right|, & \varphi \in E_l, \\ 0, & \varphi \notin E_l, \end{cases} \tag{2.4}$$

where

$$V(z)\, A(z) = \frac{P_N^*(z, \sigma_0) - P_N(z, \sigma_0)}{\mathscr{U}(z)} \in \mathbb{P}_l \tag{2.5}$$

and the polynomials A and V are such that all zeros of A are outside of Γ_{E_l} and all zeros of V are endpoints of Γ_{E_l} and the V is monic.

The singular part of σ_0 consists of at most a finite number of mass points which may appear only at zeros of A, to be more precise, at points ξ where $A(e^{i\xi})=0$.

In order to state our results on the "asymptotically periodic" measure σ, it will be useful to introduce, as in [15, Sects. 3, 4, and 5], also the following notations. Since $\mathscr{T}$, $\mathscr{U}$, and R are selfreversed polynomials

$$\begin{aligned} \tau(\varphi) &:= e^{-i(N/2)\varphi}\mathscr{T}(e^{i\varphi}), && \varphi\in[0,2\pi],\\ u(\varphi) &:= e^{-i((N-l)/2)\varphi}\mathscr{U}(e^{i\varphi}), && \varphi\in[0,2\pi],\\ \mathscr{R}(\varphi) &:= e^{-il\varphi}R(e^{i\varphi}), && \varphi\in[0,2\pi], \end{aligned} \tag{2.6}$$

are real trigonometric polynomials. Further, let

$$r(\varphi):=\begin{cases} ie^{-i(l/2)\varphi}\sqrt{R(e^{i\varphi})}=(-1)^{j+1}\sqrt{|\mathscr{R}(\varphi)|}, \\ \quad\text{for}\quad \varphi\in[\varphi_{2j-1},\varphi_{2j}], \\ e^{-i(l/2)\varphi}\sqrt{R(e^{i\varphi})}=(-1)^{j}\sqrt{|\mathscr{R}(\varphi)|}, \\ \quad\text{for}\quad \varphi\in[\varphi_{2j},\varphi_{2j+1}], \end{cases} \tag{2.7}$$

where $j=0,...,l$, with $\varphi_0:=0$ and $\varphi_{2l+1}:=2\pi$, be a real continuous square-root function, which changes sign from the interval $[\varphi_{2j-1},\varphi_{2j}]$ to the interval $[\varphi_{2j+1},\varphi_{2j+2}]$. With this notation the function f_0 from (2.4) can also be written as

$$f_0(\varphi)=\begin{cases} \dfrac{r(\varphi)}{\mathscr{V}(\varphi)\,\mathscr{A}(\varphi)}\geqslant 0, & \varphi\in E_l\\ 0, & \varphi\notin E_l \end{cases} \tag{2.8}$$

where $(\mathscr{V}\mathscr{A})(\varphi):=ie^{-i(l/2)\varphi}V(e^{i\varphi})\,A(e^{i\varphi})$ is again a real trigonometric polynomial.

We will also use the following notation: Let $z=e^{i\varphi}\in\Gamma_{E_l}$, then from (2.3), (2.6), and (2.7)

$$\left|\frac{\mathscr{T}(e^{i\varphi})+\sqrt{R(e^{i\varphi})}\,\mathscr{U}(e^{i\varphi})}{L}\right|=1,$$

hence we can define $\gamma=\gamma(\varphi)$ by

$$e^{i\gamma(\varphi)}:=e^{-i(N/2)\varphi}\frac{\mathscr{T}(e^{i\varphi})+\sqrt{R(e^{i\varphi})}\,\mathscr{U}(e^{i\varphi})}{L}, \qquad \varphi\in E_l. \tag{2.9}$$

It is easy to see, recall (2.6) and (2.7), that for $\varphi \in E_l$

$$\cos \gamma(\varphi) = \frac{\tau(\varphi)}{L} \qquad \text{and} \qquad \sin \gamma(\varphi) = -\frac{r(\varphi)\, u(\varphi)}{L}. \tag{2.10}$$

Finally, let us point out that if the reflection coefficients $a_n(\sigma)$ associated with the orthogonality measure σ satisfy condition (1.4) then the accumulation points of $\operatorname{supp}(\sigma)$ and $\operatorname{supp}(\sigma_0)$ coincide, i.e.,

$$(\operatorname{supp}(\sigma))' = (\operatorname{supp}(\sigma_0))'.$$

For $N = 1$ this fact has been proved in [4; 8, Theorem 3]. The proof given in [8] can easily be extended to the general case $N \in \mathbb{N}$. Hence, the support $\operatorname{supp}(\sigma)$, where σ denotes the perturbed measure in the sense of (1.4), also consists of the l intervals E_l and at most a denumerable number of points in $[0, 2\pi)$ outside the intervals. Moreover, the end-points of E_l, i.e., $\varphi_1, ..., \varphi_{2l}$, are the only possible accumulation points of the mass points, which all lie outside of E_l.

3. ASYMPTOTICS OF THE ORTHOGONAL POLYNOMIALS ON THE SUPPORT OF THE MEASURE OF ORTHOGONALITY

Máté, Nevai, and Totik [10, Theorem 5] have proved the following weak-$*$ limit relation:

THEOREM 1 (Máté, Nevai, and Totik [10]). *If σ satisfies*

$$\lim_{n \to \infty} P_n(0, \sigma) = 0 \tag{3.1}$$

then for any 2π-periodic Riemann integrable function g we have

$$\lim_{n \to \infty} \int_0^{2\pi} g(\varphi)\, |\Phi_n(e^{i\varphi}, \sigma)|^2\, d\sigma(\varphi) = \int_0^{2\pi} g(\varphi)\, d\varphi.$$

For measures from the Szegő-class Theorem 1 goes back to P. Turán [21] and under the assumption $\sigma' > 0$ almost everywhere on $[0, 2\pi]$ it was proved by E. A. Rahmanov [18]. Let us recall that $\sigma' > 0$ a.e. on $[0, 2\pi]$ implies, by [19, p. 106], relation (3.1).

In this section we are mainly interested in asymptotics of the orthonormal polynomials $\Phi_n(z, \sigma)$ on the support of σ. We will show how to extend Theorem 1 if assumption (3.1) is replaced by the weaker one (1.4); compare Corollary 1 below.

In [16, formula (3.33)] the authors derived the following explicit expression for the unperturbed orthonormal polynomials $\Phi_n(z, \sigma_0)$ on the arcs Γ_{E_l}: For all ν, $m \in \mathbb{N}$ and $\varphi \in E_l$ there holds

$$\begin{aligned}\Phi_{\nu N+m}(e^{i\varphi}, \sigma_0) &= \frac{iLe^{i(\nu N/2)\varphi}}{\sqrt{R(e^{i\varphi})}\,\mathcal{U}(e^{i\varphi})}[\sin(\nu\gamma(\varphi))\,\Phi_{m+N}(e^{i\varphi}, \sigma_0)\\ &\quad - e^{i(N/2)\varphi}\sin((\nu-1)\,\gamma(\varphi))\,\Phi_m(e^{i\varphi}, \sigma_0)]\\ &= \frac{e^{i((\nu-1)N/2)\varphi}}{\sin\gamma(\varphi)}[\sin(\nu\gamma(\varphi))\,\Phi_{m+N}(e^{i\varphi}, \sigma_0)\\ &\quad - e^{i(N/2)\varphi}\sin((\nu-1)\,\gamma(\varphi))\,\Phi_m(e^{i\varphi}, \sigma_0)];\end{aligned} \tag{3.2}$$

recall the definition of γ in (2.9). This formula shows the oscillating behaviour of these polynomials on the arcs with respect to ν and that, for instance, pointwise convergence can not be expected in general. But we will be able to derive weak asymptotics. Our first main results are the following two theorems.

THEOREM 2. *Suppose that* (1.4) *holds and let j and k, $j \leqslant k$, be any nonnegative integers. Then*

$$\begin{aligned}&\lim_{\nu\to\infty}\int_0^{2\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,\overline{\Phi_{\nu N+j}(e^{i\varphi}, \sigma)}\,\Phi_{\nu N+k}(e^{i\varphi}, \sigma)\,d\sigma(\varphi)\\ &\quad = \frac{2\pi L}{z^j}\left(\frac{B_{(j,k)}(z)}{\sqrt{R(z)}} + C_{(j,k)}(z)\right) - 2\pi\delta_{jk}\end{aligned} \tag{3.3}$$

uniformly on compact subsets of $\mathbb{C}\backslash\{e^{i\varphi}: \varphi \in \operatorname{supp}(\sigma)\}$, *where* $B_{(j,k)}$ *and* $C_{(j,k)}$ *are polynomials given by*

$$B_{(j,k)}(z) := \frac{1}{4\mathcal{U}(z)}(\Phi_k\Psi^*_{N+j} + \Psi_k\Phi^*_{N+j} - \Phi_{N+k}\Psi^*_j - \Psi_{N+k}\Phi^*_j)(z, \sigma_0)$$

$$C_{(j,k)}(z) := \frac{1}{2L}(\Phi_k\Psi^*_j + \Psi_k\Phi^*_j)(z, \sigma_0).$$

Remark. (a) By taking complex conjugation in (3.3) and using the simple identity $(e^{i\varphi}+z)/(e^{i\varphi}-z) = -(e^{i\varphi}+y)/(e^{i\varphi}-y)$, $y = 1/\bar{z}$, Theorem 2 can be extended easily to the case $k > j$.

(b) Let $j \leqslant k \leqslant j+N$. Then, using the identities in (1.6), the polynomials $B_{(j,k)}$ and $C_{(j,k)}$ can be represented in the form

$$B_{(j,k)}(z) = \frac{z^k}{4\mathcal{U}(z)} \left(\Phi^{(k)*}_{N+j-k}(z,\sigma_0) + \Psi^{(k)*}_{N+j-k}(z,\sigma_0)\right)$$

$$- \frac{z^j}{4\mathcal{U}(z)} \left(\Phi^{(j)}_{N+k-j}(z,\sigma_0) + \Psi^{(j)}_{N+k-j}(z,\sigma_0)\right)$$

and

$$C_{(j,k)}(z) = \frac{z^j}{2L} \left(\Phi^{(j)}_{k-j}(z,\sigma_0) + \Psi^{(j)}_{k-j}(z,\sigma_0)\right).$$

THEOREM 3. *Suppose that* (1.4) *holds and let* j *and* k, $j \leqslant k$, *be any nonnegative integers. Then*

$$\lim_{\nu \to \infty} \int_0^{2\pi} g(\varphi)\, \overline{\Phi_{\nu N+j}(e^{i\varphi},\sigma)}\, \Phi_{\nu N+k}(e^{i\varphi},\sigma)\, d\sigma(\varphi)$$

$$= \int_{E_l} g(\varphi) \frac{b_{(j,k)}(\varphi)}{r(\varphi)}\, d\varphi \tag{3.4}$$

for any 2π*-periodic Riemann integrable function* g, *where*

$$b_{(j,k)}(\varphi) := iLe^{-i(l/2+j)\varphi} B_{(j,k)}(e^{i\varphi})$$

and $B_{(j,k)}$ *given as in Theorem* 2.

If we put $k=j$ in (3.4) then we obtain the announced extension of Theorem 1:

COROLLARY 1. *Suppose that* (1.4) *is satisfied. Then for all nonnegative integers* j *there holds*

$$\lim_{\nu \to \infty} \int_0^{2\pi} g(\varphi)\, |\Phi_{\nu N+j}(e^{i\varphi},\sigma)|^2\, d\sigma(\varphi) = \int_{E_l} g(\varphi) \frac{b_{(j)}(\varphi)}{r(\varphi)}\, d\varphi \tag{3.5}$$

for any 2π*-periodic Riemann integrable function* g, *where*

$$b_{(j)}(\varphi) := b_{(j,j)}(\varphi) = \frac{ie^{-i(N/2)\varphi}}{2u(\varphi)} \left(P^{(j)*}_N + \Omega^{(j)*}_N - P^{(j)}_N - \Omega^{(j)}_N\right)(e^{i\varphi},\sigma_0)$$

is a real trigonometric polynomial.

Remark. If all the a_n^0's vanish simultaneously, i.e., $a_n \to 0$ as $n \to \infty$, then in our notation $N = l = 1$, $R(z) = (1-z)^2$, $\mathscr{U}(z) = 1$, and

$$\frac{(P_1^{(j)*} + \Omega_1^{(j)*} - P_1^{(j)} - \Omega_1^{(j)})(z, \sigma_0)}{\sqrt{R(z)}} = \frac{2-2z}{1-z} \equiv 2.$$

The limit relation in Corollary 1 becomes

$$\lim_{n\to\infty} \int_0^{2\pi} g(\varphi)\, |\Phi_n(e^{i\varphi}, \sigma)|^2\, d\sigma(\varphi) = \int_0^{2\pi} g(\varphi)\, d\varphi,$$

which is Theorem 1.

COROLLARY 2. *Suppose that* (1.4) *is satisfied. Then there holds*

$$\lim_{n\to\infty} \int_0^{2\pi} g(\varphi)\, \mathrm{Im}\{e^{-i(N/2)\varphi}\overline{\Phi_n(e^{i\varphi}, \sigma)}\, \Phi_{n+N}(e^{i\varphi}, \sigma)\}\, d\sigma(\varphi)$$
$$= \frac{1}{L} \int_{E_l} g(\varphi)\, r(\varphi)\, u(\varphi)\, d\varphi$$

and

$$\lim_{n\to\infty} \int_0^{2\pi} g(\varphi)\, \mathrm{Im}\{e^{-iN\varphi}\overline{\Phi_n(e^{i\varphi}, \sigma)}\, \Phi_{n+2N}(e^{i\varphi}, \sigma)\}\, d\sigma(\varphi)$$
$$= \frac{1}{2d_N^0} \int_{E_l} g(\varphi)\, \tau(\varphi)\, r(\varphi)\, u(\varphi)\, d\varphi$$

for any 2π*-periodic Riemann integrable function* g.

Next, let

$$K_n(z, \xi; \sigma) := \sum_{k=0}^{n} \Phi_k(z, \sigma)\, \overline{\Phi_k(\xi, \sigma)}$$

be the Christoffel function (also called reproducing kernel function) with respect to the measure σ. Then we have the following asymptotic behaviour:

THEOREM 4. *Suppose that* (1.4) *is satisfied. Then*

$$\lim_{n\to\infty} \frac{1}{n} \int_0^{2\pi} g(\varphi)\, K_n(e^{i\varphi}, e^{i\varphi}; \sigma)\, d\sigma(\varphi) = \frac{2}{N} \int_{E_l} g(\varphi)\, \frac{\tau'(\varphi)}{r(\varphi)\, u(\varphi)}\, d\varphi, \tag{3.6}$$

where τ, u, *and* r *are given as in* (2.6) *and* (2.7), *respectively, for any* 2π-*periodic Riemann integrable function* g.

Theorem 4 can also be stated in the following form:

Corollary 3. *Suppose that* (1.4) *is satisfied. Then*

$$\lim_{n\to\infty}\frac{1}{n}\int_{|\xi|=1}\tilde{g}(\xi)\,K_n(\xi,\xi;\sigma)\,d\tilde{\sigma}(\xi)=\int_{\Gamma_{E_l}}\tilde{g}(\xi)\,\mu'_{E_l}(\xi)\,d\xi, \tag{3.7}$$

where μ_{E_l} *denotes the equilibrium distribution on* Γ_{E_l}. *Here*, $\tilde{\sigma}(\xi)=\sigma(\varphi)$, $\xi=e^{i\varphi}$, *and* $\tilde{g}$ *is any Riemann integrable function on the circumference. Furthermore,*

$$\mu'_{E_l}(\xi)=\begin{cases}\dfrac{1}{\pi}\left(-\dfrac{i}{\xi}\right)\left|\dfrac{S_l(\xi)}{\sqrt{R(\xi)}}\right| & \text{for}\quad \xi\in\Gamma_{E_l},\\[2ex] 0 & \text{elsewhere},\end{cases} \tag{3.8}$$

where the polynomial $S_l\in\mathbb{P}_l$ *is defined by* $S_l(e^{i\varphi})\,\mathscr{U}(e^{i\varphi}):=(2/N)\,e^{i(N/2)\varphi}\tau'(\varphi)$. *Hence,*

$$\pi\mu'_{E_l}(\xi)\,d\xi=\frac{2}{N}\frac{\tau'(\varphi)}{r(\varphi)\,u(\varphi)}\,d\varphi,\qquad \xi=e^{i\varphi},\quad \varphi\in E_l.$$

4. PROOFS

We first prove the following lemma, which shows that we can restrict ourselves to the case of unperturbed, periodic orthonormal polynomials $\Phi_n(z,\sigma_0)$.

Lemma 1. *Let* j, k *be arbitrary nonnegative integers. Under the assumption* (1.4) *we have*

$$\lim_{n\to\infty}\left[\int_0^{2\pi}g(\varphi)\,\overline{\Phi_{n+j}(e^{i\varphi},\sigma)}\,\Phi_{n+k}(e^{i\varphi},\sigma)\,d\sigma(\varphi)\right.$$
$$\left.-\int_0^{2\pi}g(\varphi)\,\overline{\Phi_{n+j}(e^{i\varphi},\sigma_0)}\,\Phi_{n+k}(e^{i\varphi},\sigma_0)\,d\sigma_0(\varphi)\right]=0 \tag{4.1}$$

for any 2π-*periodic Riemann integrable function* g.

Proof. Let $n \in \mathbb{N}_0$ and $m \in \mathbb{Z}$. By (1.1) and (1.2) the integral

$$\int_0^{2\pi} e^{im\varphi} \overline{\Phi_n(e^{i\varphi}, \sigma)}\, \Phi_n(e^{i\varphi}, \sigma)\, d\sigma(\varphi) = \int_0^{2\pi} e^{im\varphi}\, |\Phi_n(e^{i\varphi}, \sigma)|^2\, d\sigma(\varphi)$$

can be written as a closed and continuous expression in terms of the variables $a_{n+|m|-1}, ..., a_{n-|m|}$ (set $a_j := 0$ if $j < 0$). Hence, for a fixed m we obtain from (1.4)

$$\lim_{n \to \infty} \left[\int_0^{2\pi} e^{im\varphi}\, |\Phi_n(e^{i\varphi}, \sigma)|^2\, d\sigma(\varphi) - \int_0^{2\pi} e^{im\varphi}\, |\Phi_n(e^{i\varphi}, \sigma_0)|^2\, d\sigma_0(\varphi) \right] = 0.$$

In the same way we get

$$\lim_{n \to \infty} \left[\int_0^{2\pi} e^{im\varphi}\, \overline{\Phi_n(e^{i\varphi}, \sigma)}\, \Phi_n^*(e^{i\varphi}, \sigma)\, d\sigma(\varphi) \right.$$
$$\left. - \int_0^{2\pi} e^{im\varphi}\, \overline{\Phi_n(e^{i\varphi}, \sigma_0)}\, \Phi_n^*(e^{i\varphi}, \sigma_0)\, d\sigma_0(\varphi) \right] = 0.$$

Let us assume, without loss of generality, that $k \geqslant j$. Then, by applying the recurrence relation (1.1) $(k-j)$-times, we obtain

$$\Phi_{n+k}(z, \sigma) = p_{k-j,n}(z)\, \Phi_{n+j}(z, \sigma) + q_{k-j-1,n}(z)\, \Phi^*_{n+j}(z, \sigma)$$
$$\Phi_{n+k}(z, \sigma_0) = p^0_{k-j,n}(z)\, \Phi_{n+j}(z, \sigma_0) + q^0_{k-j-1,n}(z)\, \Phi^*_{n+j}(z, \sigma_0),$$

where $p_{k-j,n}$, $p^0_{k-j,n}$ and $q_{k-j-1,n}$, $q^0_{k-j-1,n}$ are polynomials, depending on n of degree not greater than $k-j$ and $k-j-1$, respectively. By (1.4) it is easy to see that

$$\lim_{n \to \infty} (p_{k-j,n}(z) - p^0_{k-j,n}(z)) = 0$$
$$\lim_{n \to \infty} (q_{k-j-1,n}(z) - q^0_{k-j-1,n}(z)) = 0$$

uniformly on compact subsets of $\mathbb{C}$. Now the assertion (4.1) follows for all functions of the form $g(\varphi) = e^{im\varphi}$ and m an integer. But then the lemma also holds true for all trigonometric polynomials and consequently for all 2π-periodic continuous functions. Applying one-sided approximation arguments [20, Theorem 1.5.4] we obtain the assertion. ∎

Let us return to the proofs of the main results:

Proof of Theorem 2. First we show that (3.3) holds pointwise for all $z \in \mathbb{C} \setminus \{e^{i\varphi}: \varphi \in \text{supp}(\sigma)\}$. Hence, let us fix such a point z. By Lemma 1 it suffices to consider only periodic, unperturbed polynomials $\Phi_n(z, \sigma_0)$. To be on the safe side let us, for the moment, also assume that z is not a mass point of σ_0. We will see later that even this fact will not cause any problem.

The following identity holds

$$\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \overline{\Phi_{\nu N+j}(e^{i\varphi}, \sigma_0)}\, \Phi_{\nu N+k}(e^{i\varphi}, \sigma_0)\, d\sigma_0(\varphi)$$

$$= \overline{\Phi_{\nu N+j}(1/\bar{z}, \sigma_0)} \int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \Phi_{\nu N+k}(e^{i\varphi}, \sigma_0)\, d\sigma_0(\varphi)$$

$$-\int_0^{2\pi} \frac{e^{-i\varphi}+1/z}{e^{-i\varphi}-1/z} [\overline{\Phi_{\nu N+j}}(e^{-i\varphi}, \sigma_0) - \overline{\Phi_{\nu N+j}}\,(1/z, \sigma_0)]$$

$$\times \Phi_{\nu N+k}(e^{i\varphi}, \sigma_0)\, d\sigma_0(\varphi).$$

By reasons of orthogonality the second integral at the right-hand side vanishes if $k > j$ and gives 2π if $k = j$. Hence we have

$$\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \overline{\Phi_{\nu N+j}(e^{i\varphi}, \sigma_0)}\, \Phi_{\nu N+k}(e^{i\varphi}, \sigma_0)\, d\sigma_0(\varphi)$$

$$= 2\pi(z^{k-j}\Phi^*_{\nu N+j}(z, \sigma_0)\, \mathscr{G}_{\nu N+k}(z, \sigma_0) - \delta_{jk}), \tag{4.2}$$

where

$$\mathscr{G}_{\nu N+k}(z, \sigma_0) := \frac{1}{2\pi z^{\nu N+k}} \int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \Phi_{\nu N+k}(e^{i\varphi}, \sigma_0)\, d\sigma_0(\varphi) \tag{4.3}$$

denotes the function of the second kind with respect to σ_0. Introducing the polynomials

$$\mathscr{Q}_n(z, \sigma_0)\, \mathscr{U}(z) = L\Phi_{n+N}(z, \sigma_0) - \mathscr{T}(z)\, \Phi_n(z, \sigma_0), \qquad n \in \mathbb{N}_0, \tag{4.4}$$

we have the following limit-representations, cf. [16, Theorem 2.1 and relation (3.7)],

$$\lim_{\nu\to\infty} 2\Phi^*_{\nu N+j}(z, \sigma_0) \Big/ \left(\frac{\mathscr{T}(z) + \sqrt{R(z)}\, \mathscr{U}(z)}{L}\right)^{\nu}$$

$$= \left(\Phi^*_j(z, \sigma_0) + \frac{\mathscr{Q}^*_j(z, \sigma_0)}{\sqrt{R(z)}}\right)\Bigg),$$

where the convergence holds uniformly on compact subsets of $\mathbb{C}\backslash\Gamma_{E_l}$. Moreover, by [16, relation (3.7)]

$$\mathscr{G}_{\nu N+k}(z,\sigma_0)=\frac{1}{z^{\nu N+k}V(z)\,A(z)}\left(\frac{\mathscr{T}(z)-\sqrt{R(z)}\,\mathscr{U}(z)}{L}\right)^{\nu}\times(\sqrt{R(z)}\,\Phi_k(z,\sigma_0)-\mathscr{Q}_k(z,\sigma_0)). \tag{4.5}$$

Hence, using the relation $\mathscr{T}^2(z)-R(z)\,\mathscr{U}^2(z)=L^2z^N$, we obtain

$$\lim_{\nu\to\infty}\int_0^{2\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,\overline{\Phi_{\nu N+j}(e^{i\varphi},\sigma_0)}\,\Phi_{\nu N+k}(e^{i\varphi},\sigma_0)\,d\sigma_0(\varphi)$$
$$=\pi\frac{(\sqrt{R(z)}\,\Phi_j^*(z,\sigma_0)+\mathscr{Q}_j^*(z,\sigma_0))(\sqrt{R(z)}\,\Phi_k(z,\sigma_0)-\mathscr{Q}_k(z,\sigma_0))}{z^j\sqrt{R(z)}\,V(z)\,A(z)}-2\pi\delta_{jk}. \tag{4.6}$$

In order to simplify the right hand side we write

$$\begin{aligned}(\sqrt{R(z)}\,\Phi_j^*(z,\sigma_0)&+\mathscr{Q}_j^*(z,\sigma_0))(\sqrt{R(z)}\,\Phi_k(z,\sigma_0)-\mathscr{Q}_k(z,\sigma_0))\\&=(R(z)\,\Phi_k(z,\sigma_0)\,\Phi_j^*(z,\sigma_0)-\mathscr{Q}_k(z,\sigma_0)\,\mathscr{Q}_j^*(z,\sigma_0))\\&\quad-\sqrt{R(z)}(\Phi_j^*(z,\sigma_0)\,\mathscr{Q}_k(z,\sigma_0)-\Phi_k(z,\sigma_0)\,\mathscr{Q}_j^*(z,\sigma_0)).\end{aligned} \tag{4.7}$$

For further simplifications let us state a couple of identities: From (1.6) we get

$$\begin{aligned}2\Phi_k(z,\sigma_0)&=\Phi_j(z,\sigma_0)(\Phi_{k-j}^{(j)}(z,\sigma_0)+\Psi_{k-j}^{(j)}(z,\sigma_0))\\&\quad+\Phi_j^*(z,\sigma_0)(\Phi_{k-j}^{(j)}(z,\sigma_0)-\Psi_{k-j}^{(j)}(z,\sigma_0))\end{aligned} \tag{4.8}$$

and, by (4.4), (4.8), and the periodicity of the reflection coefficients $\{a_n^0\}$,

$$\begin{aligned}2\mathscr{Q}_k(z,\sigma_0)&=\mathscr{Q}_j(z,\sigma_0)(\Phi_{k-j}^{(j)}(z,\sigma_0)+\Psi_{k-j}^{(j)}(z,\sigma_0))\\&\quad+\mathscr{Q}_j^*(z,\sigma_0)(\Phi_{k-j}^{(j)}(z,\sigma_0)-\Psi_{k-j}^{(j)}(z,\sigma_0)).\end{aligned} \tag{4.9}$$

Furthermore,

$$\begin{aligned}R(z)\,\Phi_j(z,\sigma_0)\,\Phi_j^*(z,\sigma_0)-\mathscr{Q}_j(z,\sigma_0)\,\mathscr{Q}_j^*(z,\sigma_0)&=z^jV(z)\,A(z)\,H_{(j)}(z)\\R(z)\,\Phi_j^{*2}(z,\sigma_0)-\mathscr{Q}_j^{*2}(z,\sigma_0)&=z^jV(z)\,A(z)\,G_{(j)}(z),\end{aligned} \tag{4.10}$$

which follow after straightforward but tedious calculations (compare also [15, (4.5) and (4.11)]). Here, $H_{(j)}$ and $G_{(j)}$ are polynomials and are given by

$$H_{(j)}(z) := [P_N^{(j)*}(z, \sigma_0) + \Omega_N^{(j)*}(z, \sigma_0) - P_N^{(j)}(z, \sigma_0) - \Omega_N^{(j)}(z, \sigma_0)]/\mathscr{U}(z) \tag{4.11}$$

$$G_{(j)}(z) := 2(\Omega_N^{(j)*}(z, \sigma_0) - P_N^{(j)*}(z, \sigma_0))/\mathscr{U}(z). \tag{4.12}$$

Next we obtain

$$\begin{aligned}
&2(R(z)\, \Phi_k(z, \sigma_0)\, \Phi_j^*(z, \sigma_0) - \mathscr{Q}_k(z, \sigma_0)\, \mathscr{Q}_j^*(z, \sigma_0)) \\
&\quad = (R(z)\, \Phi_j(z, \sigma_0)\, \Phi_j^*(z, \sigma_0) - \mathscr{Q}_j(z, \sigma_0)\, \mathscr{Q}_j^*(z, \sigma_0)) \\
&\qquad \times (\Phi_{k-j}^{(j)}(z, \sigma_0) + \Psi_{k-j}^{(j)}(z, \sigma_0)) \\
&\qquad + (R(z)\, \Phi_j^{*2}(z, \sigma_0) - \mathscr{Q}_j^{*2}(z, \sigma_0))(\Phi_{k-j}^{(j)}(z, \sigma_0) - \Psi_{k-j}^{(j)}(z, \sigma_0)) \\
&\quad = z^j V(z)\, A(z)(H_{(j)}(z)(\Phi_{k-j}^{(j)}(z, \sigma_0) + \Psi_{k-j}^{(j)}(z, \sigma_0)) \\
&\qquad + G_{(j)}(z)(\Phi_{k-j}^{(j)}(z, \sigma_0) - \Psi_{k-j}^{(j)}(z, \sigma_0))).
\end{aligned}$$

For the second term at the right hand side of (4.7) we get, again by (4.8) and (4.9),

$$\begin{aligned}
&2(\Phi_j^*(z, \sigma_0)\, \mathscr{Q}_k(z, \sigma_0) - \Phi_k(z, \sigma_0)\, \mathscr{Q}_j^*(z, \sigma_0)) \\
&\quad = (\Phi_{k-j}^{(j)}(z, \sigma_0) + \Psi_{k-j}^{(j)}(z, \sigma_0)) \\
&\qquad \times (\Phi_j^*(z, \sigma_0)\, \mathscr{Q}_j(z, \sigma_0) - \Phi_j(z, \sigma_0)\, \mathscr{Q}_j^*(z, \sigma_0)) \\
&\quad = -2z^j V(z)\, A(z)(\Phi_{k-j}^{(j)}(z, \sigma_0) + \Psi_{k-j}^{(j)}(z, \sigma_0)).
\end{aligned}$$

For the last identity compare again [15, Sect. 4, Proof of Corollary 4.2].

Thus, the expression in (4.6) turns out to be

$$\begin{aligned}
&\lim_{\nu \to \infty} \frac{1}{\pi} \int_0^{2\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z}\, \overline{\Phi_{\nu N + j}(e^{i\varphi}, \sigma_0)}\, \Phi_{\nu N + k}(e^{i\varphi}, \sigma_0)\, d\sigma_0(\varphi) \\
&\quad = \left[\left(\frac{H_{(j)}(z)}{2\sqrt{R(z)}} + 1 \right) (\Phi_{k-j}^{(j)}(z, \sigma_0) + \Psi_{k-j}^{(j)}(z, \sigma_0)) \right. \\
&\qquad \left. + \frac{G_{(j)}(z)}{2\sqrt{R(z)}} (\Phi_{k-j}^{(j)}(z, \sigma_0) - \Psi_{k-j}^{(j)}(z, \sigma_0)) \right] - 2\delta_{jk}.
\end{aligned} \tag{4.13}$$

Finally, again by using the identities in (1.6), it can be shown that

$$H_{(j)}(z)(\Phi^{(j)}_{k-j}(z,\sigma_0)+\Psi^{(j)}_{k-j}(z,\sigma_0))+G_{(j)}(z)(\Phi^{(j)}_{k-j}(z,\sigma_0)-\Psi^{(j)}_{k-j}(z,\sigma_0))$$
$$=\frac{L}{z^j\mathcal{U}(z)}(\Phi_k\Psi^*_{N+j}+\Psi_k\Phi^*_{N+j}-\Phi_{N+k}\Psi^*_j-\Psi_{N+k}\Phi^*_j)(z,\sigma_0) \tag{4.14}$$

and

$$\Phi^{(j)}_{k-j}(z,\sigma_0)+\Psi^{(j)}_{k-j}(z,\sigma_0)=\frac{1}{z^j}(\Phi_k\Psi^*_j+\Psi_k\Phi^*_j)(z,\sigma_0). \tag{4.15}$$

This proves (3.3) pointwise for all $z\notin\{e^{i\varphi}: \varphi\in\operatorname{supp}(\sigma)\cup\operatorname{supp}(\sigma_0)\}$.

In order to get uniform convergence on compact subsets K of $\mathbb{C}\backslash\{e^{i\varphi}:\varphi\in\operatorname{supp}(\sigma)\}$ we only have to apply the Stieltjes–Vitali theorem since both sides in (3.3) are uniformly bounded functions on K. ∎

For Theorem 3, which is one of the central results of this paper, we will present two independent proofs using completely different techniques. The first proof is based on Theorem 2 and the second one on the explicit representation (3.2) of the orthogonal polynomials on the support.

First Proof of Theorem 3. Let us start with the case $N=1$ and $\Gamma_{E_1}=\{|z|=1\}$, i.e., $\lim_{n\to\infty}a_n=0$. Then we have $d\sigma_0(\varphi)=d\varphi$, $\Phi_n(z,\sigma_0)=z^n$ and by Lemma 1

$$\lim_{\nu\to\infty}\int_0^{2\pi} g(\varphi)\,\overline{\Phi_{\nu+j}(e^{i\varphi},\sigma)}\,\Phi_{\nu+k}(e^{i\varphi},\sigma)\,d\sigma(\varphi)=\int_0^{2\pi} g(\varphi)\,e^{i(k-j)\varphi}\,d\varphi,$$

which coincides with the given representation since in this special case

$$B_{(j,k)}(z)=\frac{z^k}{2}(1-z) \qquad \text{and} \qquad r(\varphi)=ie^{-i\varphi/2}(1-e^{i\varphi}).$$

Next we consider the general case $\Gamma_{E_l}\neq\{|z|=1\}$. For abbreviation let us put

$$I_{(j,k)}(z):=\frac{2\pi L}{z^j}\left(\frac{B_{(j,k)}(z)}{\sqrt{R(z)}}+C_{(j,k)}(z)\right), \qquad j\leqslant k\in\mathbb{N}_0. \tag{4.16}$$

For all $j\leqslant k$ the functions $I_{(j,k)}$ are analytic in the extended plane $\mathbb{C}$ from which Γ_c has been deleted, where

$$\Gamma_c:=\{e^{i\varphi}: \varphi\in[\varphi_1,\varphi_{2l}]\}.$$

Note that both polynomials $B_{(j,k)}$ and $C_{(j,k)}$ have a zero of order (at least) j at $z=0$, which follows from (4.14) and (4.15), respectively. In fact, $I_{(j,k)}$ is even analytic on $\mathbb{C}\backslash\Gamma_{E_l}$. By Theorem 2 we have

$$I_{(j,k)}(z) = \lim_{\nu\to\infty} \int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \overline{\Phi_{\nu N+j}(e^{i\varphi},\sigma)}\, \Phi_{\nu N+k}(e^{i\varphi},\sigma)\, d\sigma(\varphi) + 2\pi\delta_{jk}$$

and orthogonality implies

$$I_{(j,k)}(\infty) = \lim_{z\to\infty} I_{(j,k)}(z) = 0.$$

Moreover, the boundary values $I^{\pm}_{(j,k)}(\xi)$, $\xi\in\Gamma_c$, given by

$$I^{+}_{(j,k)}(\xi) = \lim_{z\to\xi,\ |z|>1} I_{(j,k)}(z)$$

$$I^{-}_{(j,k)}(\xi) = \lim_{z\to\zeta,\ |z|<1} I_{(j,k)}(z),$$

fulfill all the regularity properties of [9, Theorem 4.1 and its extension in Sect. 5]. Hence, the representation

$$I_{(j,k)}(z) = \frac{1}{2\pi i}\int_{\Gamma_c} \frac{1}{\xi - z}\,[I^{+}_{(j,k)}(\xi) - I^{-}_{(j,k)}(\xi)]\, d\xi \tag{4.17}$$

holds. Since $I_{(j,k)}$ is analytic on $\mathbb{C}\backslash\Gamma_{E_l}$ and

$$\lim_{z\to\xi,\ |z|<1} \sqrt{R(z)} =: \sqrt[-]{R(\xi)} = \sqrt[+]{R(\xi)} := -\lim_{z\to\xi,\ |z|>1} \sqrt{R(z)}$$

we have

$$I^{+}_{(j,k)}(\xi) - I^{-}_{(j,k)}(\xi) = \begin{cases} \dfrac{4\pi L}{\xi^j}\dfrac{B_{(j,k)}(\xi)}{\sqrt{R(\xi)}}, & \xi\in\Gamma_{E_l} \\ 0, & \xi\in\Gamma_c\backslash\Gamma_{E_l}, \end{cases}$$

where $\sqrt{R(\xi)} := \sqrt[+]{R(\xi)}$. Using the identity

$$\frac{1}{\xi - z} = \frac{1}{2\xi}\left[\frac{\xi+z}{\xi-z} + 1\right]$$

and the substitution $\xi = e^{i\varphi}$, (4.17) gives

$$I_{(j,k)}(z) = \int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\frac{b_{(j,k)}(\varphi)}{r(\varphi)}\, d\varphi + \int_{E_l} \frac{b_{(j,k)}(\varphi)}{r(\varphi)}\, d\varphi.$$

Moreover,

$$I_{(j,k)}(0) = 2\int_{E_l} \frac{b_{(j,k)}(\varphi)}{r(\varphi)}\, d\varphi = \begin{cases} 0, & \text{for } j \neq k \\ 4\pi, & \text{for } j = k, \end{cases}$$

where the second identity follows from (4.16), (3.3), and the orthogonality property. Hence,

$$\int_{E_l} \frac{b_{(j,k)}(\varphi)}{r(\varphi)}\, d\varphi = 2\pi\delta_{jk}$$

and

$$I_{(j,k)}(z) = \int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \frac{b_{(j,k)}(\varphi)}{r(\varphi)}\, d\varphi + 2\pi\delta_{jk}.$$

Comparing this last representation with (4.16) and (3.3) yields

$$\lim_{\nu\to\infty} \int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \overline{\Phi_{\nu N+j}(e^{i\varphi},\sigma)}\, \Phi_{\nu N+k}(e^{i\varphi},\sigma)\, d\sigma(\varphi)$$

$$= \int_{E_l} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \frac{b_{(j,k)}(\varphi)}{r(\varphi)}\, d\varphi. \tag{4.18}$$

The above identity holds for all $z \in \mathbb{C} \setminus \{e^{i\varphi}\colon \varphi \in \operatorname{supp}(\sigma)\}$. By expanding

$$\frac{e^{i\varphi}+z}{e^{i\varphi}-z} = 1 + 2\sum_{n=1}^{\infty} e^{-in\varphi} z^n \qquad \text{for} \quad |z| < 1$$

and

$$\frac{e^{i\varphi}+z}{e^{i\varphi}-z} = -\frac{e^{-i\varphi}+1/z}{e^{-i\varphi}-1/z} = -1 - 2\sum_{n=1}^{\infty} e^{in\varphi} y^n, \qquad y := \frac{1}{z}, \quad |z| > 1,$$

and comparing coefficients in (4.18), assertion (3.4) follows for all functions $g(\varphi) = e^{im\varphi}$, $m \in \mathbb{Z}$. Now the same arguments as at the end of the proof of Lemma 1 give the result. ∎

Let us sketch the second variant of proof which does not make use of Theorem 2.

Second Proof of Theorem 3. By Lemma 1 it suffices only to consider the periodic orthonormal polynomials $\Phi_n(z, \sigma_0)$. In the following all the orthonormal polynomials will correspond to the measure σ_0 and we simply write $\Phi_n(z)$, $\Psi_n(z)$, etc.

From (3.2) we get

$$\sin^2(\gamma(\varphi))\,\overline{\Phi_{\nu N+j}(e^{i\varphi})}\,\Phi_{\nu N+k}(e^{i\varphi})$$
$$=\sin^2(\nu\gamma(\varphi))\,\Phi_{N+k}(e^{i\varphi})\,\overline{\Phi_{N+j}(e^{i\varphi})}$$
$$+\sin^2((\nu-1)\,\gamma(\varphi))\,\Phi_k(e^{i\varphi})\,\overline{\Phi_j(e^{i\varphi})}$$
$$-\sin((\nu-1)\,\gamma(\varphi))\sin(\nu\gamma(\varphi))$$
$$\times(e^{i(N/2)\,\varphi}\Phi_k(e^{i\varphi})\,\overline{\Phi_{N+j}(e^{i\varphi})}+e^{-i(N/2)\,\varphi}\Phi_{N+k}(e^{i\varphi})\,\overline{\Phi_j(e^{i\varphi})}).$$

Let us recall that γ is strictly monotone and differentiable on each of the intervals $[\varphi_{2j-1},\varphi_{2j}]$, $j=1,...,l$, and maps all these intervals onto $[0,\pi]$. Furthermore it is known that

$$\lim_{\nu\to\infty}\int_0^\pi g(x)\sin^2(\nu x)\,dx=\tfrac{1}{2}\int_0^\pi g(x)\,dx \tag{4.19}$$

and

$$\lim_{\nu\to\infty}\int_0^\pi g(x)\sin(\nu x)\sin((\nu-1)\,x)\,dx=\tfrac{1}{2}\int_0^\pi g(x)\cos x\,dx \tag{4.20}$$

for every 2π-periodic Riemann integrable function g. Now, by transforming the variable of integration in (3.4) from φ to $\gamma=\gamma(\varphi)$, then letting ν tend to infinity, applying (4.19) and (4.20) and transforming back, it is not difficult to see that

$$\sin^2(\gamma(\varphi))\,\overline{\Phi_{\nu N+j}(e^{i\varphi})}\,\Phi_{\nu N+k}(e^{i\varphi})$$
$$\xrightarrow[\nu\to\infty]{*}\Phi_{N+k}(e^{i\varphi})\,\overline{\Phi_{N+j}(e^{i\varphi})}$$
$$+\Phi_k(e^{i\varphi})\,\overline{\Phi_j(e^{i}\varphi)}$$
$$-\cos\gamma(\varphi)(e^{i(N/2)\,\varphi}\Phi_k(e^{i\varphi})\,\overline{\Phi_{N+j}(e^{i\varphi})}$$
$$+e^{-i(N/2)\,\varphi}\Phi_{N+k}(e^{i\varphi})\,\overline{\Phi_j(e^{i\varphi})}) \tag{4.21}$$

on the set E_l for the class of all 2π-periodic Riemann integrable functions. By the definition of the reversed polynomials and by (2.10), the right hand side is nothing but, $z=e^{i\varphi}$,

$$\frac{1}{2z^{N+j}}\Bigg[\Phi_{N+k}(z)\,\Phi^*_{N+j}(z)+z^N\Phi_k(z)\,\Phi^*_j(z)$$
$$-\frac{1}{L}\mathscr{T}(z)(\Phi_k(z)\,\Phi^*_{N+j}(z)+\Phi_{N+k}(z)\,\Phi^*_j(z))\Bigg].$$

Tedious but straightforward calculations, using the identities in (1.6), give that this last expression can also be written as

$$\frac{V(z)\,A(z)\,\mathcal{U}(z)}{4Lz^{N+j}}\,(\Phi_{N+k}(z)\,\Psi_j^*(z)+\Psi_{N+k}(z)\,\Phi_j^*(z)$$
$$-\Phi_k(z)\,\Psi_{N+j}^*(z)-\Psi_k(z)\,\Phi_{N+j}^*(z)).$$

Now we make use of (2.8) and (2.10), i.e.,

$$\mathscr{V}(\varphi)\,\mathscr{A}(\varphi)=ie^{-i(l/2)\,\varphi}\,V(e^{i\varphi})\,A(e^{i\varphi})$$
$$\sin\gamma(\varphi)=-\frac{r(\varphi)\,u(\varphi)}{L},\qquad \varphi\in E_l$$
$$\sigma_0'(\varphi)=\frac{r(\varphi)}{\mathscr{V}(\varphi)\,\mathscr{A}(\varphi)},\qquad \varphi\in E_l,$$

then (4.21) takes the form

$$\overline{\Phi_{\nu N+j}(e^{i\varphi})}\,\Phi_{\nu N+k}(e^{i\varphi})\,\sigma_0'(\varphi)$$
$$\xrightarrow[\nu\to\infty]{*}\frac{iL}{4z^{l/2+j}r(\varphi)\,u(\varphi)}$$
$$\times(\Phi_k\Psi_{N+j}^*+\Psi_k\Phi_{N+j}-\Phi_{N+k}\Psi_j^*-\Psi_{N+k}\Phi_j^*)(z),$$
$$z=e^{i\varphi},\tag{4.22}$$

again on the set E_l for the class of all 2π-periodic Riemann integrable functions.

Recall that the singular component of the measure σ_0 consists of at most finitely many mass points, which are located outside the intervals E_l, and that σ_0 is absolutely continuous on E_l. At the mass points the orthonormal polynomials tend to zero as n goes to infinity, since at these points the sum $\sum_{n=1}^{\infty}|\Phi_n(z)|^2$ exists. Hence, (4.22) already gives the desired assertion. ∎

Proof of Corollary 1. The statement (3.5) is an immediate consequence of Theorem 3, the Remark after Theorem 2, and the identities (2.3) and (2.6). ∎

Proof of Corollary 2. In a similar way as in the proof of Lemma 1 it can be shown that it suffices to show the limit relations only for the "periodic" measure σ_0. To show the statement for the measure σ_0, as usual, we consider trigonometric polynomials first, then 2π-periodic continuous functions, and finally 2π-periodic Riemann integrable functions.

One way to prove the corollary is to apply Theorem 3. But this causes some calculations. A shorter way is to proceed as follows. Obviously, for every $n \in \mathbb{N}$,

$$\int_0^{2\pi} g(\varphi) \operatorname{Im}\{e^{-i(N/2)\varphi} \overline{\Phi_n(e^{i\varphi}, \sigma_0)}\, \Phi_{n+N}(e^{i\varphi}, \sigma_0)\}\, d\sigma_0(\varphi)$$

$$= \frac{i}{2} \int_0^{2\pi} g(\varphi)[e^{i(N/2)\varphi} \Phi_n(e^{i\varphi}, \sigma_0)\, \overline{\Phi_{n+N}(e^{i\varphi}, \sigma_0)}$$

$$- e^{-i(N/2)\varphi} \overline{\Phi_n(e^{i\varphi}, \sigma_0)}\, \Phi_{n+N}(e^{i\varphi}, \sigma_0)]\, d\sigma_0(\varphi)$$

$$= \frac{i}{2} \int_0^{2\pi} g(\varphi)\, e^{-i(n+N/2)\varphi}[\Phi_n(e^{i\varphi}, \sigma_0)\, \Phi^*_{n+N}(e^{i\varphi}, \sigma_0)$$

$$- \Phi^*_n(e^{i\varphi}, \sigma_0)\, \Phi_{n+N}(e^{i\varphi}, \sigma_0)]\, d\sigma_0(\varphi).$$

Because of the periodicity of the reflection coefficients it can be shown by straightforward calculations that

$$\Phi_n(z, \sigma_0)\, \Phi^*_{n+N}(z, \sigma_0) - \Phi^*_n(z, \sigma_0)\, \Phi_{n+N}(z, \sigma_0)$$

$$= z^n(\Phi^*_N(z, \sigma_0) - \Phi_N(z, \sigma_0)) = \frac{z^n V(z)\, A(z)\, \mathscr{U}(z)}{\sqrt{d^0_N}}.$$

Since, by (2.8) resp. by (2.4) and the lines thereafter,

$$d\sigma_0(\varphi) = \frac{r(\varphi)\, d\varphi}{i e^{-i(l/2)\varphi} V(e^{i\varphi})\, A(e^{i\varphi})} + \text{possibly point measures at zeros of } A$$

the first assertion follows; recall that $2\sqrt{d^0_N} = L$.

By the periodicity of the reflection coefficients $a^0_n = a^0_{n+N} = a^0_{n+2N}$ for $n \in \mathbb{N}_0$ one can derive

$$\Phi_n(z, \sigma_0)\, \Phi^*_{n+2N}(z, \sigma_0) - \Phi^*_n(z, \sigma_0)\, \Phi_{n+2N}(z, \sigma_0)$$

$$= z^n(\Phi^*_{2N}(z, \sigma_0) - \Phi_{2N}(z, \sigma_0))$$

$$= \frac{z^n}{d^0_N} (P^*_{2N}(z, \sigma_0) - P_{2N}(z, \sigma_0))$$

$$= \frac{z^n}{d^0_N} \mathscr{T}(z)\, V(z)\, A(z)\, \mathscr{U}(z)$$

by straightforward calculation. Now the second assertion can be proved as the first part of the corollary. ∎

Proof of Theorem 4. From Corollary 1 we obtain by routine calculation

$$\lim_{n\to\infty} \frac{1}{n} \int_0^{2\pi} g(\varphi)\, K_n(e^{i\varphi}, e^{i\varphi}; \sigma)\, d\sigma(\varphi)$$
$$= \frac{1}{N} \int_{E_l} g(\varphi) \sum_{j=0}^{N-1} \frac{b_{(j)}(\varphi)}{r(\varphi)}\, d\varphi. \tag{4.23}$$

Combining the definition of $b_{(j)}$ in Corollary 1 and (4.11) we have

$$b_{(j)}(\varphi) = \frac{i}{2}\, e^{-i(l/2)\,\varphi} H_{(j)}(e^{i\varphi}).$$

Hence, we are looking for a closed expression for the sum

$$\sum_{j=0}^{N-1} \frac{b_{(j)}(\varphi)}{r(\varphi)} = \sum_{j=0}^{N-1} \frac{H_{(j)}(e^{i\varphi})}{2\sqrt{R(e^{i\varphi})}} = \lim_{s\to 1^-} \sum_{j=0}^{N-1} \frac{H_{(j)}(se^{i\varphi})}{2\sqrt{R(se^{i\varphi})}}, \qquad \varphi \in E_l.$$

From (4.10) we know

$$R(z)\, \Phi_j(z, \sigma_0)\, \Phi_j^*(z, \sigma_0) - \mathcal{Q}_j(z, \sigma_0)\, \mathcal{Q}_j^*(z, \sigma_0)$$
$$= z^j V(z)\, A(z)\, H_{(j)}(z), \tag{4.24}$$

where by (4.4)

$$\mathcal{Q}_j(z, \sigma_0)\, \mathcal{U}(z) = L\Phi_{j+N}(z, \sigma_0) - \mathcal{T}(z)\, \Phi_j(z, \sigma_0).$$

Let us define the functions

$$\begin{aligned} \Phi_j^{\pm}(z) &:= \Phi_{j+N}(z, \sigma_0) - y^{\pm}(z)\, \Phi_j(z, \sigma_0) \\ \Phi_j^{*\pm}(z) &:= \Phi_{j+N}^*(z, \sigma_0) - y^{\pm}(z)\, \Phi_j^*(z, \sigma_0) \end{aligned} \tag{4.25}$$

with

$$y^{\pm}(z) := \frac{\mathcal{T}(z) \pm \sqrt{R(z)}\, \mathcal{U}(z)}{L}.$$

Then the above identity (4.24) can be written in the form

$$\Phi_j^{+}(z)\, \overline{\Phi_j^{-}(1/\bar{z})} + \Phi_j^{-}(z)\, \overline{\Phi_j^{+}(1/\bar{z})}$$
$$= -\frac{2}{L^2 z^N}\, V(z)\, A(z)\, \mathcal{U}^2(z)\, H_{(j)}(z). \tag{4.26}$$

By definition (4.25) and by the periodicity of the reflection coefficients $\{a_n^0 = a_{n+N}^0\}$ it is not difficult to see that the functions $\Phi_n^{\pm}$ and $\Phi_n^{*\,\pm}$ fulfill the same recurrence-relation as the orthonormal polynomials $\Phi_n(z, \sigma_0)$, that is,

$$\sqrt{1-|a_n^0|^2}\,\Phi_{n+1}^{\pm}(z) = z\Phi_n^{\pm}(z) + a_n^0 \Phi_n^{*\,\pm}(z)$$
$$\sqrt{1-|a_n^0|^2}\,\Phi_{n+1}^{*\,\pm}(z) = \Phi_n^{*\,\pm}(z) + z\bar{a}_n^0 \Phi_n^{\pm}(z).$$

Further, by [15, Corollary 4.1(a)] we have

$$\Phi_{2N}(z, \sigma_0) = \frac{2\mathscr{T}(z)}{L}\,\Phi_N(z, \sigma_0) - z^N,$$

and thus

$$\Phi_N^{\pm}(z) = y^{\mp}(z)\,\Phi_0^{\pm}(z) \tag{4.27}$$
$$\Phi_N^{*\,\pm}(z) = y^{\mp}(z)\,\Phi_0^{*\,\pm}(z).$$

Now we can use quite the same techniques as for the proof of the Christoffel–Darboux formula (cf., e.g., [7, Chap. 1.1.4, p. 8]) to obtain

$$\sum_{j=0}^{n} \Phi_j^{\pm}(z)\,\overline{\Phi_j^{\pm}(\xi)} = \frac{\begin{pmatrix} \Phi_{n+1}^{*\,\pm}(z)\,\overline{\Phi_{n+1}^{*\,\mp}(\xi)} - \Phi_{n+1}^{\pm}(z)\,\overline{\Phi_{n+1}^{\mp}(\xi)} \\ -\Phi_0^{*\,\pm}(z)\,\overline{\Phi_0^{*\,\mp}(\xi)} + \Phi_0^{\pm}(z)\,\overline{\Phi_0^{\mp}(\xi)} \end{pmatrix}}{1 - z\bar{\xi}}$$

Taking $n = N-1$ and using the initial-property (4.27) gives

$$\sum_{j=0}^{N-1} \Phi_j^{\pm}(z)\,\overline{\Phi_j^{\mp}(\xi)} = \frac{y^{\mp}(z)\,\overline{y^{\pm}(\xi)}}{1 - z\bar{\xi}}\,[\Phi_0^{*\,\pm}(z)\,\overline{\Phi_0^{*\,\mp}(\xi)} - \Phi_0^{\pm}(z)\,\overline{\Phi_0^{\mp}(\xi)}]. \tag{4.28}$$

Since we are interested to get a closed expression for the sum over the $H_{(j)}$'s we have, by (4.26), to evaluate the above identity at $\xi = 1/\bar{z}$. Using the selfreversed-property of the polynomials $\mathscr{T}$, $\mathscr{U}$, and R we obtain for all $z \in \mathbb{C}$

$$\overline{y^{\pm}(1/\bar{z})} = \frac{y^{\pm}(z)}{z^N},$$

which gives

$$y^{\mp}(z)\,\overline{y^{\pm}(1/\bar{z})}=\frac{y^{\mp}(z)\,y^{\pm}(z)}{z^N}=\frac{\mathcal{T}^2(z)-R(z)\,\mathcal{U}^2(z)}{L^2z^N}\equiv 1. \tag{4.29}$$

Let us recall in this connection that for $z\in\Gamma_{E_l}$ we have by (2.7) (see [14, Lemma 3.1 and Remark 3.2])

$$\sqrt[+]{R(e^{i\varphi})}=-\sqrt[-]{R(e^{i\varphi})}=-\sqrt{R(e^{i\varphi})}=-ie^{i(l/2)\,\varphi}r(\varphi), \tag{4.30}$$

where, as usual,

$$\sqrt[+]{R(e^{i\varphi})}:=\lim_{s\to 1^+}\sqrt{R(se^{i\varphi})}\qquad\text{and}\qquad \sqrt[-]{R(e^{i\varphi})}:=\lim_{s\to 1^-}\sqrt{R(se^{i\varphi})}.$$

In order to calculate the sum in (4.28) we use the following limiting processes. First let ξ tend to $e^{i\varphi}$ from the interior of the unit disk and z to $e^{i\psi}$ from the exterior of the unit disk, and then let $\psi\to\varphi$.

Let us consider the right hand side of (4.28) in order to calculate

$$\sum_{j=0}^{N-1}\Phi_j^{\pm}(e^{i\varphi})\,\overline{\Phi_j^{\mp}(e^{i\varphi})}=\lim_{\psi\to\varphi}\sum_{j=0}^{N-1}\Phi_j^{\pm}(e^{i\psi})\,\overline{\Phi_j^{\mp}(e^{i\varphi})}.$$

By (4.29) and de l'Hospital's rule we get

$$\begin{aligned}
&\lim_{\psi\to\varphi}\frac{y^{\mp}(e^{i\psi})\,\overline{y^{\pm}(e^{i\varphi})}-1}{1-e^{i(\psi-\varphi)}}\\
&\quad=\lim_{\psi\to\varphi}\frac{e^{iN(\psi-\varphi)/2}\{(\tau(\psi)\mp ir(\psi)\,u(\psi))(\tau(\varphi)\pm ir(\varphi)\,u(\varphi))\}-L^2}{L^2(1-e^{i(\psi-\varphi)})}\\
&\quad=-\frac{N}{2}-\frac{[\tau'(\varphi)\mp i(r(\varphi)\,u(\varphi))'][\tau(\varphi)\pm ir(\varphi)]}{iL^2}.
\end{aligned}$$

Taking the limit process $z\to e^{i\psi}$, $|z|>1$, $\xi\to e^{i\varphi}$, $|\xi|<1$, and $\psi\to\varphi$, we obtain by (4.30) that

$$\begin{aligned}
&\Phi_0^{*\pm}(e^{i\varphi})\,\overline{\Phi_0^{*\mp}(e^{i\varphi})}-\Phi_0^{\pm}(e^{i\varphi})\,\overline{\Phi_0^{\mp}(e^{i\varphi})}\\
&\quad=\frac{2\sqrt{R(e^{i\varphi})}\,\mathcal{U}(e^{i\varphi})}{Le^{iN\varphi}}[\Phi_N^*(e^{i\varphi},\sigma_0)-\Phi_N(e^{i\varphi},\sigma_0)].
\end{aligned}$$

Now by (4.26), (4.28), and the above representations we get

$$
\begin{aligned}
&-\frac{2}{L^2 e^{iN\varphi}}\, V(e^{i\varphi})\, A(e^{i\varphi}) \sum_{j=0}^{N-1} H_{(j)}(e^{i\varphi}) \\
&\quad= \frac{2\sqrt{R(e^{i\varphi})}\, \mathscr{U}(e^{i\varphi})}{iL^3 e^{iN\varphi}} \left[\Phi_N^*(e^{i\varphi}, \sigma_0) - \Phi_N(e^{i\varphi}, \sigma_0)\right] \\
&\qquad \times ([\tau'(\varphi) + i(r(\varphi)\, u(\varphi))'][\tau(\varphi) - ir(\varphi)\, u(\varphi)] \\
&\qquad - [\tau'(\varphi) - i(r(\varphi)\, u(\varphi))'][\tau(\varphi) + ir(\varphi)\, u(\varphi)]) \\
&\quad= \frac{4\sqrt{R(e^{i\varphi})}\, \mathscr{U}(e^{i\varphi})}{L^3 e^{iN\varphi}} \left[\Phi_N^*(e^{i\varphi}, \sigma_0) - \Phi_N(e^{i\varphi}, \sigma_0)\right] \\
&\qquad \times ((r(\varphi)\, u(\varphi))'\, \tau(\varphi) - r(\varphi)\, u(\varphi)\, \tau'(\varphi)),
\end{aligned}
$$

and by $V(z)\, A(z)\, \mathscr{U}(z) = \sqrt{d_N^0}(\Phi_N^*(z, \sigma_0) - \Phi_N(z, \sigma_0))$

$$
\sum_{j=0}^{N-1} \frac{b_{(j)}(\varphi)}{r(\varphi)} = \frac{1}{2d_N^0}\, (r(\varphi)\, u(\varphi)\, \tau'(\varphi) - (r(\varphi)\, u(\varphi))'\, \tau(\varphi)).
$$

Finally, let us show the identity

$$
r(\varphi)\, u(\varphi)\, \tau'(\varphi) - (r(\varphi)\, u(\varphi))'\, \tau(\varphi) = \frac{4d_N^0 \tau'(\varphi)}{r(\varphi)\, u(\varphi)}. \tag{4.31}
$$

Then the desired relation (3.6) follows from (4.23).

From the second line in (2.3) and from (2.7) we obtain

$$
\tau^2(\varphi) + r^2(\varphi)\, u^2(\varphi) = L^2, \qquad \varphi \in E_l,
$$

i.e.,

$$
r^2(\varphi)\, u^2(\varphi) = L^2 - \tau^2(\varphi) \quad \text{and} \quad r(\varphi)\, u(\varphi)(r(\varphi)\, u(\varphi))' = -\tau(\varphi)\, \tau'(\varphi).
$$

Substituting these identities in the left hand side of (4.31) gives

$$
\begin{aligned}
&r(\varphi)\, u(\varphi)\, \tau'(\varphi) - (r(\varphi)\, u(\varphi))'\, \tau(\varphi) \\
&\quad= \frac{1}{r(\varphi)\, u(\varphi)}\, ((L^2 - \tau^2(\varphi))\, \tau'(\varphi) + \tau^2(\varphi)\, \tau'(\varphi)) \\
&\quad= \frac{L^2 \tau'(\varphi)}{r(\varphi)\, u(\varphi)} = \frac{4d_N^0 \tau'(\varphi)}{r(\varphi)\, u(\varphi)}.
\end{aligned}
$$

This finishes the proof. ∎

Proof of Corollary 3. We will follow some ideas, given by the first author in [12, Lemma 2.2(a)]. In [17, relation (5.12)] we have shown that the complex Green function $G(z, \infty) =: G(z)$ of $\overline{\mathbb{C}} \backslash \Gamma_{E_l}$ with pole at ∞ is given by

$$G(z) = \frac{1}{2} \int_{z_1}^{z} \frac{1}{\xi} \left(1 - \frac{iS_l(\xi)}{\sqrt{R(\xi)}}\right) d\xi, \qquad z_1 = e^{i\varphi_1}, \tag{4.32}$$

where the polynomial S_l is given by $S_l(e^{i\varphi})\, \mathscr{U}(e^{i\varphi}) := (2/N)\, e^{i(N/2)\varphi} \tau'(\varphi)$ and where the integration is performed along a path in the complex plane cut along Γ_{E_l}. On the other hand we have the representation (cf. [23, Sect. 14])

$$G(z) = \int_{\Gamma_{E_l}} \ln(z - \xi)\, d\mu_{E_l}(\xi) - \ln \gamma(\Gamma_{E_l}), \tag{4.33}$$

where μ_{E_l} is the equilibrium distribution on Γ_{E_l} and where $\gamma(\Gamma_{E_l})$ denotes the capacity of Γ_{E_l}. Differentiating the identities in (4.32) and (4.33) gives

$$\int_{\Gamma_{E_l}} \frac{1}{z - \xi}\, d\mu_{E_l}(\xi) = \frac{1}{2z} \left(1 - \frac{iS_l(z)}{\sqrt{R(z)}}\right) =: \Phi(z).$$

With the help of the Sochozki–Plemelj formulas (see, e.g., [11, 9; Theorem 4.1]) we obtain

$$2\pi i \mu'_{E_l}(\xi) = \Phi^+(\xi) - \Phi^-(\xi) = 2i \left(-\frac{i}{\xi}\right) \left|\frac{S_l(\xi)}{\sqrt{R(\xi)}}\right|, \qquad \xi \in \Gamma_{E_l},$$

which is (3.8). Now (3.7) immediately follows from Theorem 4. ∎

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